NOTATION

$g(\tau)$, heat flux at surface of plate; $t(x, \tau)$ temperature at the point x; $n(x, \tau)$, noise of temperature measurement; $\tilde{t}(x, \tau)$, temperature measured at the point x; τ, time; $\tilde{t}$ and g, vectors composed of the values of the functions $\tilde{t}(x, \tau)$ and $g(\tau)$; H, matrix approximating the original integral equation; α, regularization parameter.

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NUMERICAL ALGORITHM FOR THE SOLUTION OF LINEAR TWO-DIMENSIONAL INTEGRAL EQUATIONS OF THE FIRST KIND

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A numerical algorithm is proposed for the solution of two-dimensional integral equations of the first kind, to which some inverse problems of heat conduction are reduced.

It is known that many problems of practical importance in the analysis of experimental results lead to the solution of a linear integral equation of the first kind in a rectangular region (for example, thermophysical problems of the determination of the heat flux to axisymmetric and plane bodies from the assigned time dependence of the temperature at part of the boundary of the region, the geophysical problem of the interface between two media with different densities, and others):

$$Af \equiv \iint_{D_1} K(x, y, s, t) f(s, t)\, ds dt = \psi(x, y), \tag{1}$$

$$x, y \in D = \{0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1\}; \quad s, t \in D_1(x, y) \subset D.$$

The problem of the solution of such an equation is, generally speaking, incorrectly stated. If the solution of Eq. (1) is unique for an assigned right side $\psi(x, y)$ then the solution $f(s, t)$ can be obtained by the regularization method proposed by Tikhonov [1, 2]. In accordance with this method an approximate solution $f^{\alpha}(s, t)$ is defined as a function yielding the minimum of the functional

$$M^{\alpha}[f, \psi] \equiv \iint_D [Af - \psi]^2\, dxdy + \alpha \iint_D \{p_1 f^2 + p_2 f_s^2 + p_3 f_t^2 + \beta [p_4 f_{ss}^2 + p_5 f_{tt}^2 + p_6 f_{st}^2]\}\, dsdt, \tag{2}$$

in which the value of the regularization parameter α must conform with the level of the root-mean-square error δ of the right side. In the functional (2) the quantities $p_i(s, t) > 0$ ($i = 1, 2, \ldots, 6$) are assigned functions and $\beta = 0$ or 1 in first- or second-order regularization, respectively.

In [3] an algorithm was proposed for the solution of the variational problem (2) for the one-dimensional equation (1), based on the approximation of the unknown solution by cubic splines [4]. The effectiveness of the algorithm, verified on the problem of solving the Abel equation [3] and the problem of reconstructing a distribution function [5], is explained to a considerable extent by the properties of the convergence of cubic splines

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(with a decrease in the interpolation step the values of the spline and of its first and second derivatives converge uniformly to the values of the approximated function and its derivatives [4]).

In the present paper this algorithm is applied to linear two-dimensional integral equations of the first kind.

Let us partition the unit square into $(N-1)^2$ small squares by the lines $s_i = h(i-1)$ and $t_j = h(j-1)$, $(h = 1/(N-1))$, $i, j = 1, 2, \ldots, N$) and define the N fundamental splines $A_i(s)$ by the following relations:

$$A_i(s_j) = \delta_{ij} \ (i, j = 1, 2, \ldots, N), \quad A_i'(s_j) = 0 \ (i = 1, 2, \ldots, N; \ j = 1, N). \tag{3}$$

Then the function $f(s, t)$, which satisfies the conditions

1. $f_s = 0$ at the nodes P_{ij} $(i = 1, N; j = 1, 2, \ldots, N)$,
2. $f_t = 0$ at the nodes P_{ij} $(i = 1, 2, \ldots, N; j = 1, N)$,
3. $f_{st} = 0$ at the nodes P_{ij} $(i = 1, N; j = 1, N)$

(4)

at the boundary of the region, can be approximated by the doubly cubic spline [4]

$$f(s, t) \approx S(f; s, t) = \sum_{i,j=1}^{N} A_i(s) A_j(t) f(s_i, t_j). \tag{5}$$

Substituting this expression for $f(s, t)$ into the functional (2), we have

$$\begin{aligned} M^{\alpha}[f, \psi] = {} & \sum_{i,j,m,n=1}^{N} f_{ij} f_{mn} \int_0^1 \int_0^1 J_{ij}(x, y) J_{mn}(x, y) \, dxdy \\ & - 2 \sum_{i,j=1}^{N} f_{ij} \int_0^1 \int_0^1 \psi(x, y) J_{ij}(x, y) \, dxdy + \int_0^1 \int_0^1 [\psi(x, y)]^2 \, dxdy \\ & + \alpha \sum_{i,j,m,n=1}^{N} f_{ij} f_{mn} \int_0^1 \int_0^1 \{A_i(s) A_j(t) A_m(s) A_n(t) \\ & + A_i'(s) A_j(t) A_m'(s) A_n(t) + A_i(s) A_j'(t) A_m(s) A_n'(t) \\ & + \beta [A_i''(s) A_j(t) A_m''(s) A_n(t) + A_i(s) A_j''(t) A_m(s) A_n''(t) \\ & + A_i'(s) A_j'(t) A_m'(s) A_n'(t)]\} \, dsdt, \end{aligned} \tag{6}$$

where

$$J_{ij}(x, y) = \iint_{D_t} K(x, y, s, t) A_i(s) A_j(t) \, dsdt. \tag{7}$$

Now, if one uses the quadrature equation with the coefficients B_{pq} and the same system of nodes to calculate the outer integral over the rectangular region, and one uses the definition (3), then the functional (6) is converted to the form

$$M^{\alpha}[f, \psi] = \sum_{i,j,m,n=1}^{N} (\lambda_{ijmn}^{(1)} + \alpha \lambda_{ijmn}^{(2)}) f_{ij} f_{mn} - 2 \sum_{i,j=1}^{N} \mu_{ij} f_{ij} + c, \tag{8}$$

$$\lambda_{ijmn}^{(1)} = \sum_{p,q=1}^{N} B_{pq} J_{ij}(x_p, y_q) J_{mn}(x_p, y_q),$$

$$\begin{aligned} \lambda_{ijmn}^{(2)} = {} & \sum_{p,q=1}^{N} B_{pq} \{\delta_{ip}\delta_{jq}\delta_{mp}\delta_{nq} + [A_i'(s_p) A_m'(s_p) \\ & + \beta A_i''(t_q) A_m''(s_p)] \delta_{jq}\delta_{nq} + [A_j'(t_q) A_n'(t_q) + \beta A_j''(t_q) A_n''(t_q)] \delta_{ip}\delta_{mp} \\ & + \beta A_i'(s_p) A_j'(t_q) A_m'(s_p) A_n'(t_q)\}, \end{aligned}$$

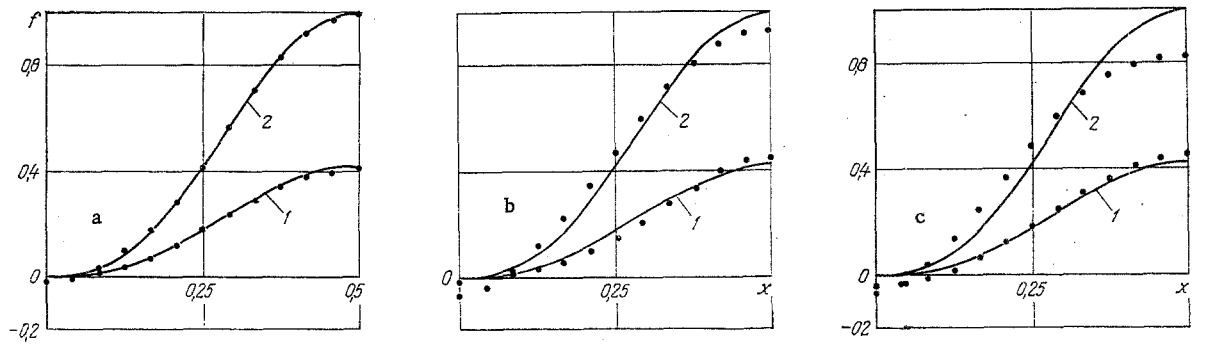

Fig. 1. Dependence of the solution f on x with y = 0.25 (1) and y = 0.5 (2); a) $\Delta = 0$; $\alpha_{opt} = 10^{-11}$; b) $\Delta = 0.002$; $\alpha_{opt} = 10^{-8}$; c) $\Delta = 0.01$; $\alpha_{opt} = 3.16 \cdot 10^{-7}$.

$$\mu_{ij} = \sum_{p,q=1}^{N} B_{pq} J_{ij}(x_p, y_q)\psi(x_p, y_q),$$

$$c = \sum_{p,q=1}^{N} B_{pq} [\psi(x_p, y_q)]^2.$$

From (6) and (8) it is easy to see that $\lambda^{(1)}_{ijmn} = \lambda^{(1)}_{mnij}$ and $\lambda^{(2)}_{ijmn} = \lambda^{(2)}_{mnij}$. Now we introduce a new numbering of the nodes in the unit square, connected with the old one by the relations $k = N(i - 1) + j$ and $l = N(m - 1) + n$ ($k, l = 1, 2, \ldots, N^2$). Then

$$M^{\alpha}[f, \psi] = \sum_{k,l=1}^{N^2} \lambda_{kl} f_k f_l - 2 \sum_{k=1}^{N^2} \mu_k f_k + c, \tag{9}$$

where $\lambda_{kl} = \lambda^{(1)}_{kl} + \alpha\lambda^{(2)}_{kl}$ is a symmetric matrix. The condition of steadiness of the functional (9) has the form

$$\sum_{k=1}^{N^2} \lambda_{kl} f_k = \mu_l \quad (l = 1, 2, \ldots, N^2), \tag{10}$$

i.e., the problem of finding the minimum of the functional (2) is reduced to the problem of solving a system of N^2 linear algebraic equations with a symmetric matrix of coefficients.

Let us dwell on some of the possibilities and computational properties of the proposed algorithm.

1. The algorithm is universal in the class of equations of the type of (1) in the sense that using it one can solve integral equations of the first kind for which the kernel has a weak singularity.

2. The algorithm provides a second order of regularization (with $\beta = 1$).

3. It is clear that in the realization of the algorithm the greater part of the computer time must be expended on the calculation of the integrals $J_{ij}(x_p, y_q)$. In this connection we note that since $A_i(s)$ are piecewise-cubic functions, for a rather broad class of kernels these integrals can be calculated in quadratures. (Such a situation occurs, in particular, in many inverse problems of heat conduction where the kernel of the integral equation consists of a series, each term of which contains functions of the types $\sin\pi mx$, $\cos\pi mx$, and $\exp[-\pi^2 m^2(t - \tau)]$. In these problems the calculation of $J_{ij}(x_p, y_q)$ comes down to the calculation of the sum of a rapidly converging series.) If this cannot be done, then for the given kernel and a fixed grid these integrals (and consequently the coefficients $\lambda^{(1)}_{kl}$ and $\lambda^{(2)}_{kl}$) are calculated once and can be stored in external computer memories. Then for each right side (new experiment) one calculates only the N^2 coefficients μ_k. From what has been said it follows that in a mass calculation of variants most of the time will be expended in the repeated solution of the system of equations (10) with different values of the regularization parameter. In the process the matrix elements λ_{kl} for each α are calculated from the elements $\lambda^{(1)}_{kl}$ and $\lambda^{(2)}_{kl}$ of the stored arrays by the formula $\lambda_{kl} = \lambda^{(1)}_{kl} + \alpha\lambda^{(2)}_{kl}$.

4. The fact that the matrix of coefficients λ_{kl} is symmetrical for any α makes it possible to store only the upper or lower triangular parts of the matrix. For large enough N this procedure allows one to reduce the required volume of the internal computer memory by almost half (or, with a given volume of internal memory, to increase the number N).

The proposed algorithm was realized in the form of a program in the FORTRAN language and tested on the model problem of the solution of the integral equation

$$\int_0^1\int_0^1 f(s,\ t)\,[(s-x)^2+(t-y)^2+1]^{-3/2}\,ds dt = \psi(x,\ y). \tag{11}$$

The geophysical problem of determining the surface $f(x, y)$ separating two media of different densities from the data of gravimetric measurements $\psi(x, y)$ at the earth's surface is reduced to an equation of this type [6, 7].

The test of the correctness and efficiency of operation of the program was carried out as follows: Using (11) the right side $\psi(x, y)$ was calculated with the maximum possible accuracy for $f(s, t) = 4096(s - s^2)^3(t - t^2)^3$ and then from it $f(x, y)$ was reconstructed using the described algorithm.

In the solution of this problem it was assumed that $p_i = 1$ ($i = 1, 2, \ldots, 6$) and regularization of the first and second orders was used. The errors inherent to the obtainment of the experimental data were modeled by imposing a random disturbance onto $\psi(x, y)$; i.e., instead of $\psi(x, y)$ we assigned the function $\psi^*(x, y) = \psi(x, y) + \eta\Delta$ in the calculations, where η is a random quantity uniformly distributed in the interval $[-1, 1]$ and Δ is some constant multiplier. The calculations were carried out with $N = 9$ and $N = 13$ ($N = 13$ is close to the extreme possible number of nodes over each of the variables when only the internal computer memory is used to store the matrix of coefficients λ_{kl}). The regularization parameter α was chosen by the method of the quasibest approximation [2, 7].

The results of the calculations in the form of the dependence $f(x, y)$ for $N = 13$, $\beta = 1$, and $\Delta = 0$, 0.002, and 0.01 are presented in Fig. 1a, b, and c. In this figure the solid curves are the exact solution of the problem, while the points are the results of the numerical solution using the proposed algorithm. With $\Delta = 0$ (exact right side) the maximum error in the reconstruction of $f(x, y)$ occurs at the boundary of the region and comprises 1-2% of $\max\limits_{x,y\in D} f(x, y)$. With $\Delta = 0.002$ and 0.01 (the error in $\psi(x, y)$ comprises ~1 and ~5% of the maximum value of $\psi(x, y)$, respectively) the errors in the reconstruction of $f(x, y)$ for different sequences of random numbers η are equal to 5-7 and 15-18% of $\max\limits_{x,y\in D} f(x, y)$. The accuracy of the reconstruction of $f(x, y)$ decreases somewhat with $N = 9$ or $\beta = 1$ (first-order regularization).

On the whole, the analysis of the results of the methodical calculations performed allows one to conclude that the proposed algorithm for the solution of equations of the type of (1) is workable and efficient.

NOTATION

$K(x, y, s, t)$, kernel of the integral equation; $f(s, t)$, unknown solution of integral equation; $\psi(x, y)$, right side of integral equation; D, unit square; $M^\alpha[f, \psi]$, Tikhonov's regularizing functional; α, regularization parameter; $p_i(x, y)$ ($i = 1, 2, \ldots, 6$), assigned positive functions; β, parameter of order of regularization; N, number of nodes over each of the variables x and y; h, step of the grid; $A_i(s)$ ($s = 1, 2, \ldots, N$), fundamental cubic splines; λ_{kl} ($k, l = 1, 2, \ldots, N^2$), coefficients of a symmetric matrix; μ_k ($k = 1, 2, \ldots, N^2$), vector components of the right sides of the system of linear equations.

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